2-local isometries on differentiable function spaces

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joint work with L. Li, C.-W. Liu and L. Wang

2021 TMS Annual Meeting

The classical Banach-Stone Theorem

Let X, Y be compact Hausdorff spaces and $T : C(X) \to C(Y)$ be a surjective linear isometry, then there exist a homeomorphism

$$\varphi : Y \to X \quad \text{and} \quad h \in C(Y)$$

such that

$$|h(y)| = 1 \qquad \text{for all } y \in Y,$$

and

$$(Tf)(y) = h(y)f(\varphi(y)) \quad \text{for all } f \in C(X) \text{ and } y \in Y.$$

- many extensions to a variety of different settings

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X, Y: Banach spaces
$\mathcal{S} \subset L(X, Y)$, all linear maps from X to Y.

A linear mapping $T : X \to Y$ is a local $\mathcal{S}$ map if for each $x \in X$, there exists $T_x \in \mathcal{S}$, depending on x satisfying

$$T(x) = T_x(x).$$

Kadison, 1990

$\mathcal{R}$: von Neumann algebra

Then every norm-continuous local derivation $T : \mathcal{R} \to \mathcal{R}$ is a derivation.

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E, F: Banach spaces
$\mathcal{S} \subset L(E, F)$, all linear maps from E to F

A (non-necessarily linear nor continuous) mapping $\Delta \colon E \to F$ is a 2-local $\mathcal{S}$-map if for any $x, y \in E$, there exists $T_{x,y} \in \mathcal{S}$, depending on x and y such that

$$\Delta(x) = T_{x,y}(x) \text{ and } \Delta(y) = T_{x,y}(y).$$

P. Šemrl, 1997

H: infinite-dimensional separable Hilbert space
$B(H)$: algebra of all linear bounded operators on H

Then every 2-local automorphism $\Delta : B(H) \to B(H)$ is an automorphism.

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Then every 2-local isometry $\Delta : C_0(X) \to C_0(X)$ is a surjective linear isometry.

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Q: Is any 2-local $\mathcal{S}$-map $\Delta : E \to E$ linear? NO.

Let $E = span\{I, E_{12}, E_{13}\} \subset M_3(\mathbb{R})$ and

$$\Delta(aI + bE_{12} + cE_{13}) = aI + bE_{12} + \sqrt[3]{b^3 + c^3}\, E_{13}.$$

Then Δ is nonlinear 2-local automorphism. Let $A = a_1 I + b_1 E_{12} + c_1 E_{13}$ and $B = a_2 I + b_2 E_{12} + c_2 E_{13}$. Note that $\exists q, s \in \mathbb{R}$ s.t.

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Then $T_{A,B}(aI + bE_{12} + cE_{13}) = aI + bE_{12} + (bq + cs)E_{13}$ is an automorphism on E and $\Delta(A) = T_{A,B}(A)$, $\Delta(B) = T_{A,B}(B)$.

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Q: Is any 2-local $\mathcal{S}$-map $\Delta\colon E \to E$ surjective? No.

Győry, 2001

X: uncountable space with discrete topology
Y: proper subset of X for which there is a bijection $\psi : X \to Y$

Define the map $\Delta : C_0(X) \to C_0(X)$ by

$$\Delta(f)(x) = \begin{cases} f(\psi^{-1}(x)) & \text{if } x \in Y \\ 0 & \text{if } x \in X \setminus Y. \end{cases}$$

Then Δ is a linear 2-local isometry which is not surjective.

$f \in C_0(X)$ iff $\exists x_n$ s.t. $f(x_n) \to 0$ and supp$f \subseteq \{x_n\}$

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$C^1([0,1])$: continuously diff functions on $[0,1]$, with the norm

$$||f|| = \sup_{x\in[0,1]} \{|f(x)| + |f'(x)|\}$$

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M. Hosseini, 2017

$C^{(n)}([0,1])$: n-times cont diff on $[0,1]$ with the norm

$$\|f\|_n = \max\{|f(0)|, |f'(0)|, \ldots, |f^{(n-1)}(0)|, \|f^{(n)}\|_\infty\}$$

$T : C^{(n)}([0,1]) \to C^{(n)}([0,1])$ 2-local real-linear isometry, i.e., for $f, g \in C^{(n)}([0,1])$, $\exists T_{f,g} : C^{(n)}([0,1]) \to C^{(n)}([0,1])$ onto real-linear isometry, s.t. $T(f) = T_{f,g}(f)$ and $T(g) = T_{f,g}(g)$.

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$C^1([0,1])$ with the norm $||f|| = \sup_{x\in[0,1]}\{|f(x)| + |f'(x)|\}$
$T : C^1([0,1]) \to C^1([0,1])$ linear surjective isometry

Then $\exists\varphi : [0,1] \to [0,1]$ with $\varphi(x) = x$ *or* $1 - x$, and a constant $e^{i\lambda}$, $\lambda \in [-\pi, \pi]$, s.t.

$$T(f)(y) = e^{i\lambda} f(\varphi(y)), \quad \forall y \in [0,1].$$

Kowalski-Słodkowski theorem, 1980

A: complex Banach algebra; $\Delta : A \to \mathbb{C}$ mapping satisfying

$$\Delta(0) = 0 \text{ and } \Delta(x) - \Delta(y) \in \sigma(x - y) \ \forall x, y \in A$$

Then Δ is linear and multiplicative.

$\sigma(a)$: the spectrum of $a \in A$

spherical variant of the Kowalski-Słodkowski theorem

A: unital complex Banach algebra; $\Delta : A \to \mathbb{C}$ be a mapping satisfying the following properties:

(a) Δ is 1-homogeneous;

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Then Δ is linear, and $\exists \lambda_0 \in \mathbb{T}$ s.t. $\lambda_0 \Delta$ is multiplicative.

$\Delta : X \to Y$ between complex Banach spaces is 1-*homogeneous* if $\Delta(\lambda x) = \lambda \Delta(x)$, for every $x \in X$, $\lambda \in \mathbb{C}$.

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Theorem

2-local isometry $\Delta : (C^1[0,1], \|\cdot\|) \to (C^1[0,1], \|\cdot\|)$ is linear

- $T : C^{(1)}[0,1] \to C^{(1)}[0,1]$ onto linear, then $\exists \tau \in \mathbb{T}$, $\varphi : [0,1] \to [0,1]$ bij. s.t.

$$T(f)(s) = \tau f(\varphi(s)) \quad \forall f \in C^1([0,1])\, s \in [0,1]$$

- $s \in [0,1]$, $f, g \in C^1$, $\exists\ \tau_{f,g,s} \in \mathbb{T}$, $\varphi_{f,g,s} : [0,1] \to [0,1]$ s.t.

$$\delta_s \circ \Delta(f) = \delta_s \circ T_{f,g,s}(f) = \tau_{f,g,s} f(\varphi_{f,g,s}(s))$$

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- $\delta_s \circ \Delta(\lambda f) = \lambda \delta_s \circ \Delta(f)$ and $\delta_s \circ \Delta$ is 1-homogeneous
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spherical variant of the Kowalski-Słodkowski theorem

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Then Δ is linear, and $\exists \lambda_0 \in \mathbb{T}$ s.t. $\lambda_0 \Delta$ is multiplicative.

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2-local isometry $\Delta : (C^1[0, 1], \| \cdot \|) \to (C^1[0, 1], \| \cdot \|)$ is linear

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